RE-3607/200

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M. A. / M. Sc. (Part - II) Examination

April / May – 2010

Mathematics : Paper - 501

(Advanced Functional Analysis)

Time : 3 Hours] [Total Marks : 70

Instrucitons :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fillup strictly the details of ☛ signs on your answer book.	Seat No. :
Name of the Examination : ☛ M. A. / M. Sc. - 2	
Name of the Subject : ☛ 501 - MATHEMATICS	
☛ Subject Code No. : 3 6 0 7 ☛ Section No. (1, 2,.....) : NIL	Student's Signature

(2) All all questions.

(3) Each question carries 14 marks.

(4) Follow usual notations.

1(*a*) Let X and Y be normed spaces and $T : X \to Y$ be a bounded linear operator. Then prove that adjoint operator $T^{\times}$ is linear, bounded and $\| T^{\times} \| = \| T \|$.

(*b*) Show that a bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping.

(*c*) Show that a sublinear functional p satisfies $p(0) = 0$ and $p(-x) \geq -p(x)$.

OR

1(*a*) State and prove the Hahn-Banach theorem for the normed spaces.

(*b*) Show that in finite dimensional normed spaces the distinction between weak convergence and strong convergence disappears.

(*c*) Let X denote the set of all real-valued functions x on the interval $[0,1]$, and let $x \leq y$ mean that $x(t) \leq y(t)$ for all $t \in [0,1]$. Show that this defines a partial ordering. Is it a total ordering? Does X have maximal elements?

2(*a*) Prove that the spectrum set $\sigma(T)$ of a bounded linear operator T on a complex Banach space X is closed.

(*b*) Give an illustration of an operator T having spectral value which is not an eigenvalue. Justify your claim.

(c) Let A be a matrix given by $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Determine $(A - \lambda I)^{-1}$.

OR

2(a) If T is a bounded linear operator on a complex Banach space, then prove that for the spectral radius $r_\sigma(T)$ of T we have $r_\sigma(T) = \lim_{n \to \infty} \sqrt[n]{\| T^n \|}$.

(b) If X is a complex Banach space, $S, T \in B(X, X)$ and $ST = TS$, then prove that

$$r_\sigma(ST) \leq r_\sigma(S) r_\sigma(T)$$

Can we drop $ST = TS$? Why ?

(c) Prove that the spectrum of a linear operator on a finite dimensional space is pure point spectrum.

3(a) Let (T_n) be a sequence of compact linear operators from a normed space X into a Banach space Y. Prove that if (T_n) is uniformly operator convergent, then the limit operator T is compact.

(b) Show that a compact linear operator $T : X \to Y$ from a normed space X into a Banach space Y has a compact linear extension $\tilde{T} : \hat{X} \to Y$, where $\hat{X}$ is the completion of X.

(c) Prove that $T : l^2 \to l^2$ defined by $Tx = y = (\eta_j)$, where $\eta_j = \xi_j / 2^j$ is compact.

OR

3(a) Show that the range of a compact linear operator $T : X \to Y$ is separable, where X and Y are normed spaces.

(b) The set of eigenvalues of a compact linear operator $T : X \to Y$ on a normed space X is countable and the only possible point of accumulation is $\lambda = 0$.

(c) Let H be a Hilbert space, $T : H \to H$ a bounded linear operator and T^* the Hilbert-adjoint operator of T. Show that T is compact if and only if T^*T is compact.

4(a) Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Show that $\lambda \in \rho(T) \Leftrightarrow$ there exists $c > 0$ such that for every $x \in H$,

$$\| T_\lambda x \| \geq c \| x \|.$$

(b) Show that the spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert space H is real.

(c) Let $T : H \to H$ and $W : H \to H$ be a bounded linear operators on a complex Hilbert space H and $S = W^* T W$. Show that if T is self-adjoint and positive, so is S.

OR

4(a) Define a projection operator and prove that a bounded linear operator $P: H \to H$ on a Hilbert space H is a projection iff P is self-adjoint and idempotent.

(b) Let P_1 and P_2 be projections on a Hilbert space H. Then show that $P = P_1 + P_2$ is a projection $H \Leftrightarrow Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ are orthogonal.

(c) Let P_1 and P_2 be projections on a Hilbert space H onto Y_1 and Y_2 respectively, and $P_1P_2 = P_2P_1$. Show that

$$P_1 + P_2 - P_1P_2$$

is a projection of H onto $Y_1 + Y_2$.

5(a) Let $T : D(T) \to H$ be a linear operator which is densely defined in a complex Hilbert space H. Suppose T is injective and its range $R(T)$ is dense in H. Then show that T^* is injective and $(T^*)^{-1} = (T^{-1})^*$.

(b) Let H be a complex Hilbert space and $T : D(T) \to H$ linear and densely defined in H. Then show that T is symmetric $\Leftrightarrow \langle Tx, x \rangle$ is real for all $x \in D(T)$.

(c) If T is symmetric show that T^{**} is symmetric.

OR

5(a) Let $T : D(T) \to H$ be a symmetric linear operator which is densely defined in a complex Hilbert space H and $\overline{T}$ be its closure then prove that $(\overline{T})^* = T^*$.

(b) Prove that a densely defined linear operator T in a complex Hilbert space H is symmetric if and only if $T \subset T^*$.

(c) Show that the multiplication operator has no eigenvalue.